Kinematics of small scale anisotropy in turbulence

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A simple, empirically motivated model is proposed to explain the transfer of imposed large scale anisotropy to small scales in high-Reynolds-number turbulence. Observed power-law scalings of anisotropy metrics are interpreted as manifestations of power-law scalings governing high-gradient regions resulting from compressional eddy motions. The model is used to interpret the measured moment-order dependencies of the exponents and amplitudes of odd-order structure functions and derivative moments that vanish in the absence of anisotropy.

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I. INTRODUCTION

The persistence of small scale anisotropy in high-Reynolds-number turbulence subject to imposed large scale anisotropy has been demonstrated experimentally [1,2], and various formal approaches to the analysis of this phenomenon have been proposed [3,4]. The goal of the present study is to propose a simple intuitive picture of the mechanism causing persistent anisotropy that accounts for some heretofore unexplained trends and features indicated by measurements.

To illustrate the physical picture on which the present analysis is based, amplification of high moments of property derivatives due to length scale reduction by turbulent eddies is considered. Assume an initially linear scalar profile, gradient G, extending a distance L in the gradient direction. Suppose that a sequence of compressional and rotational motions converts this profile into a sawtooth shape consisting of $2n+1$ segments, each of width $L/(2n+1)$ (thus filling the original interval L), with gradients alternating between $(2n+1)G$ and $-(2n+1)G$. Then the mean value of the jth derivative moment (taking the directional derivative parallel to the imposed gradient) is $[1/(2n+1)][(2n+1)G]^j$ for j odd. For $j\geqslant 3$, it is clear from this result that length scale reduction can amplify the chosen measure of small scale anisotropy despite the fact that rotational motions that flip the sign of the directional derivative result in sign cancellation over most of the domain. (In this example, n positive-slope contributions are cancelled by the n equal-and-opposite negative-slope regions. The non-null outcome is due to the one additional positive-slope region.)

The example suggests that it may be straightforward to estimate measures of small scale anisotropy by using well-established scaling laws of inertial-range turbulence to quantify the distribution of scales. However, high moments of property derivatives are sensitive to rare occurrences of extreme length scale reduction, so inertial-range scaling laws that govern typical behavior may not contain the needed information.

A further complication is the collective behavior induced by the coupling between advective and dissipative processes. Though dissipative processes are ostensibly irrelevant in the inertial range, measurements and numerical simulations indicate that turbulence and other multiscale advection-diffusion processes induce steplike profiles of properties that are subject to imposed large scale gradients [5]. Apparently, localized episodes of intense mixing homogenize some fluid regions sufficiently to expel the imposed gradient into the gaps between regions of intense mixing. The physics and the statistics of this mechanism of gradient concentration and amplification are not well understood. This precludes first-principles analysis of the origin and properties of small scale anisotropy.

An important feature of observed anisotropy is its conformance to scaling. Namely, measures of anisotropy exhibit power-law parameter dependencies that imply the absence of a distinguished length scale other than the turbulence integral scale L and a dissipative cutoff λ. Accordingly, power-law scalings are assumed in the model proposed here.

II. MODELING APPROACH

Nonvanishing odd-order inertial-range structure functions and odd derivative moments at high Reynolds number (Re) indicate the persistence of anisotropy at scales far below L. Here, these two statistical measures of anisotropy are interpreted as manifestations of a single underlying mechanism.

A model of the statistical properties of a one-dimensional (1D) profile of an advected property (velocity, passive scalar, etc.) in the direction of the imposed gradient is formulated as follows. As an idealization, the high-gradient regions of the profile are represented as a collection of individual zones, each of which is characterized by a size r and a property difference A across the zone. The property derivative within the zone is estimated as A/r, which neglects the possible influence of fine structure within the zone.

A may be positive or negative, and as indicated by the illustrative example, the tendency of vortical motions to induce isotropy implies a nearly symmetric distribution of A values. The asymmetry due to the imposed gradient (here assumed positive) is represented here as a greater number density of positive-A zones than negative-A zones.

The following simplifying assumptions are introduced. First, the magnitude of A is taken to be the same for all high-gradient zones of size r, and a dependence

$$|A(r)|=CGL(r/L)^s h(r/\lambda) \qquad (2.1)$$

is assumed. The exponent s and the coefficient C are model parameters, G is the imposed property gradient, which is an imposed shear in the experiments that are compared to the model, and h is a cutoff function that vanishes for small values of its argument and asymptotes to unity for large values of its argument. It is therefore included in estimates involving A only when r of order λ is considered. The dissipation scale λ that is applicable to high-gradient zones is not the same as the Kolmogorov microscale η. (The two scales are compared quantitatively in Sec. V.) $|A(L)|$ is not required to equal GL because A characterizes special subregions of the flow rather than the entire flow. Therefore C need not be equal to unity. Second, if the fraction of the 1D profile that is covered by zones in the size range $[r,r+dr]$ is denoted $f_+(r)dr$ for positive A and $f_-(r)dr$ for negative A, then the difference $f\equiv f_+-f_-$ is assumed to be non-negative for all r and to obey the scaling

$$f(r)\sim(1/L)(r/L)^p \tag{2.2}$$

for $\lambda<r<L$.

The Re dependence of λ will be analyzed in order to determine the Re dependence of derivative moments. For this purpose, an additional modeling assumption is needed, as follows.

A time scale t is associated with high-gradient zones of size r, based on the assumed relation

$$t=T(r/L)^q, \tag{2.3}$$

where T is a large eddy time scale. For a homogeneous shear flow with an imposed velocity gradient G, T is inversely proportional to G. The time scale t represents the formation time of size-r zones. A balance between t and the time scales of processes that dissipate high-gradient zones is used to estimate λ in Sec. IV.

For present purposes, these assumptions define the gross structure and evolution of high-gradient zones. (Some additional assumptions are introduced in the analyses of structure functions and derivative moments.) The model, which involves three unknown exponents p, q, and s as well as several multiplicative coefficients, does not address the difficult question of the physics determining their values. Rather, the present goal is to determine the relationships between these parameters and statistical measures of anisotropy, thereby providing a framework for physical interpretation of experimental results.

III. STRUCTURE FUNCTIONS

The inertial-range scalings of odd-order structure functions are analyzed based on the foregoing assumptions. Specifically, the transverse structure functions

$$D_j(R)=\langle[u(y+R)-u(y)]^j\rangle \tag{3.1}$$

are considered, where y is the coordinate parallel to the imposed mean shear and u is the velocity component normal to y to which the shear is applied. The analysis is likewise applicable to structure functions of a passive scalar.

It is assumed that high-gradient zones, as defined here, dominate the scaling of the structure functions of interest. The zone-size subranges $r<R$ and $r>R$ are considered individually.

It is assumed that any high-gradient zone of size $r<R$ that is intersected by a size-R line segment is entirely contained within the segment. The velocity difference across the segment, to leading order in scaling, is the sum of the velocity differences across the high-gradient zones contained in the segment.

For a given zone-size range $[r,r+dr]$, the net contribution to this velocity difference is estimated as $A(r)$ times the expected value of the excess number of positive-A zones in this size range that are contained in the size-R segment. The latter quantity is R times the number density $(1/r)f(r)dr$. This gives the net contribution $(R/r)A(r)f(r)dr$, which scales with r as $r^{p+s-1}dr$. Integration of this quantity over $\lambda<r<R$ gives a result that is dominated by the large-r (order R) contribution if $p+s>0$ or by the small-r (order λ) contribution if $p+s<0$. Based on the empirical observation that $D_j(R)$ scales with R rather than a dissipative length scale, it is assumed that $p+s>0$. Exponent values determined empirically in Sec. V are consistent with this assumption.

It would appear therefore that the subrange $r<R$ contributes a term of order $(R/L)^{j(p+s+1)}$ to the scaling of $D_j(R)$. In fact, this is an underestimate. The mean value of the excess number of positive-A order-R zones contained in a size-R segment scales as $Rf(R)\sim(R/L)^{p+1}$. Provided that $R\ll L$ and $p>-1$ (which is required so that the dr integral of f is finite in the limit of vanishing λ), this is much less than unity. Therefore a distinction must be made between the infrequent occurrence of a positive-A excess, which is the dominant contribution to $D_j(R)$, and the more likely absence of an excess, in which case the contribution to $D_j(R)$ is deemed negligible. Accordingly, the $r<R$ contribution to $D_j(R)$ is estimated as $A^j(R)\sim(R/L)^{js}$ times the probability that a size-R segment contains a greater number of positive-A order-R zones than negative-A order-R zones (specifically one more, because larger excesses occur with geometrically decreasing probability). This probability scales as $Rf(R)$. Combining these estimates, the $r<R$ contribution to $D_j(R)$ is found to be of order $(R/L)^{js+p+1}$, which dominates $(R/L)^{j(p+s+1)}$ for $j>1$ and $p>-1$.

If the size-R line segment is intersected by a high-gradient zone of size $r>R$, it is assumed that the segment is entirely contained within the size-r zone and that the variation of u across the zone is linear. Based on these assumptions, the velocity difference across the size-R segment is $(R/r)A(r/L)$.

For $r\gg R$, the probability that the size-R segment intersects (i.e., is contained within) a zone whose width is in the range $[r,r+dr]$ is $f_+(r)dr$ for positive A and $f_-(r)dr$ for negative A. The difference between these probabilities, $f(r)dr$, is taken to be the probability of a net contribution to the odd-order structure functions by high-gradient zones in the indicated size range.

The $r>R$ contribution to the scaling of $D_j(R)$ is, therefore, estimated to be of order $\int_R^L (R/r)^j A^j(r/L) f(r) dr$. Based on Eqs. (2.1) and (2.2), this is proportional to $C^j \int_{R/L}^1 (R/r)^j (r/L)^{js} (r/L)^p \, d(r/L)$. The integral scales as $\{1/[j(s-1)+p+1]\}(R/L)^j[1-(R/L)^{j(s-1)+p+1}]$. For $s<1-[(1+p)/j]$, this gives the scaling $(R/L)^{js+p+1}$, the same as obtained for the $r<R$ contribution. The exponent values determined empirically in Sec. V obey this inequality for j values of interest. If the inequality is not obeyed, then the amplitude falloff with decreasing r is so fast that the integral is dominated by order-L zone sizes. The assumption of linear variation of u across a high-gradient zone, and other model assumptions, break down at this length scale, so the model is applicable only if the stated equality is obeyed.

It is concluded that the ranges $r<R$ and $r>R$ obey the same scaling, which is not surprising because both ranges are dominated by the order-R contribution. The scaling

$$D_j(R) \sim C^j (R/L)^{js+p+1} \tag{3.2}$$

is thus obtained. The C^j prefactor is obtained for the $r<R$ as well as the $r>R$ contribution, although it was not shown explicitly in the analysis of the case $r<R$. The factor GL appearing in Eq. (2.1) is omitted here because it is assumed to be absorbed in the normalization of u.

A key simplification in this analysis is the assumption that all high-gradient zones of given size r have the same amplitude A, given by Eq. (2.1). The neglected fluctuations of A for given r may have significant impact on high-order structure functions. Implications with regard to comparison of the model to experimental data are discussed in Sec. V.

IV. DERIVATIVE MOMENTS

The Re dependence of odd derivative moments is analyzed. To introduce Re effects, the dissipative cutoff scale λ is estimated by comparing the time scale t of Eq. (2.3) to the time scales of processes that may break down or dissipate high-gradient zones. Two such processes are considered: eddy diffusivity (governed by Kolmogorov scaling) and viscous dissipation.

Kolmogorov scaling implies $r^{2/3}$ dependence of the eddy time scale on the eddy size r. Therefore, eddy breakdown either suppresses the small scale events entirely (if $q<\frac{2}{3}$) or is too slow to affect them at all (if $q>\frac{2}{3}$). The latter is the physically interesting case in the present context. The implied constraint, $q>\frac{2}{3}$, is discussed with regard to experimental results in Sec. V.

In view of the irrelevance of eddy breakdown in the regime of interest, λ is determined by invoking the diffusive scaling $\lambda^2 \sim \nu t \sim \nu T(\lambda/L)^q$. Henceforth, lengths and times are scaled by L and T respectively, so in scaled units, $\nu = \mathrm{Re}^{-1}$, giving

$$\lambda \sim \mathrm{Re}^{-1/(2-q)}. \tag{4.1}$$

The singularity at $q=2$ reflects the absence of a balance mechanism for $q \geqslant 2$ because length scale reduction outruns viscous dissipation for all $r<1$.

For $q=0$, the familiar Taylor-microscale scaling is recovered. This scaling is not relevant to the present situation due to the restriction $q>\frac{2}{3}$. The present physical picture is thus fundamentally different from the anisotropy analysis of Gonzalez [4], who postulates that the Taylor microscale is the cutoff scale for high-gradient events.

The jth derivative moment is estimated by multiplying the jth power of the derivative $A(r)/r$ by the measure $f(r)dr$ of the excess positive-A contribution and integrating over $\lambda<r<1$, giving

$$m_j \sim \lambda^{j(s-1)+p+1} \sim \mathrm{Re}^{[j(1-s)-p-1]/(2-q)}. \tag{4.2}$$

Owing to the dominance of the dissipation scale contribution to the integral, and the influence of the cutoff function h in Eq. (2.1) at that scale, the j dependence of the coefficient of Re in this scaling cannot be deduced. As noted in Sec. III, the model is valid only if $j(s-1)+p+1<0$.

The normalized moments of interest are $n_j \equiv m_j / m_2^{j/2}$. To evaluate these, it is necessary to determine whether the anomalous scaling, Eq. (4.2), dominates the conventional scaling for $j=2$, which is $m_2 \sim \mathrm{Re}$. The conventional scaling dominates if $q<1+p+2s$.

Evaluation of p, q, and s based on comparison to measurements (Sec. V) indicates that the conventional scaling dominates, consistent with the measured Re dependence of m_2. Therefore, the final result is

$$n_j \sim \mathrm{Re}^{[-(p+1)/(2-q)]+[(j/2)(q-2s)/(2-q)]}. \tag{4.3}$$

V. COMPARISON TO MEASUREMENTS

The experimental observation that motivates the introduction of the amplitude factor A is the dependence of the odd-order inertial-range structure-function exponents on moment order j. Equation (3.2) allows linear dependence on j. Measurements indicate slower-than-linear increase of the exponents as a function of j [1]. The measured departure from linearity may not be statistically significant, but exponents for even and odd j appear to fall on a single curve [2], in which case the well-established sublinearity for even j applies also to odd j. The departure from linearity is slight, so the available odd-order data can be used to determine the model exponent s that controls the j dependence of the structure-function exponents. Based on the measured [1] exponent values 1.1, 1.5, and 1.8 for $j=3$, 5, and 7, respectively, $s=0.2$ is a reasonable choice.

To within experimental precision, measured derivative moments exhibit linear dependence of the exponent of Re in Eq. (4.3) on moment order j [1]. Based on the choice $s=0.2$, the value of q that best matches the measured moment-order dependence is 0.75.

With these choices, p can be chosen to match the exponent sequence for either the structure functions or the derivative-moment Re dependencies. The latter is chosen because the structure-function analysis involves more significant omissions of physically relevant processes than does the derivative-moment analysis. On this basis, p is assigned the value -0.15. This choice satisfies the condition $p+s>0$ invoked in the structure-function analysis in Sec. III.

For these parameter values, the model gives R_λ exponent values -0.52, 0.04, and 0.60 for derivative-moment order $j=3$, 5, and 7, respectively. Exponents representing the dependence on $R_\lambda = \mathrm{Re}^{1/2}$ are given because the experimental results are parameterized by R_λ. The corresponding measured exponent values are -0.52, -0.02, and 0.63 [1]. The good agreement reflects the adjustment of p and q to match the linear dependence of the exponent sequence on j.

For these parameter values, the predicted structure-function exponents for $j=3$, 5, and 7 are 1.45, 1.85, and 2.25, respectively. The corresponding measured values are 1.1, 1.5, and 1.8. Physical mechanisms omitted from the model that may account for the difference between predicted and measured values are discussed in Sec. III. For $j=3$ and 5, the predicted exponents exceed the values $j/3$ that correspond to inertial-range cascade scaling with no intermittency. The predicted values, therefore, correspond to intermittency that is decreasing rather than increasing with decreasing length scale.

Clearly, the model is not a quantitatively accurate representation of small scale fluctuation mechanisms. The model may be more accurate in principle for derivative moments than for structure functions due to the local nature of the former, in contrast to the nonlocal nature of the latter. However, the structure-function analysis has been used to obtain a quantity that affects the derivative-moment scaling exponents, so quantitative inferences based on these exponents are likewise problematic. Quantitative implications of the inferred exponent values, though tentative for this reason, are nevertheless examined in order to provide a baseline for future investigation and refinement.

In Sec. IV, it is noted that $q > \frac{2}{3}$ is required so that eddy breakdown does not suppress the formation of high-gradient zones, and $q < 1 + p + 2s$ is required to match the experimentally observed scaling of m_2. For the empirically determined values of p and s, this implies $q < 1.25$. The empirical value $q = 0.75$ obeys these constraints. It is interesting to note that this value of q implies $\lambda \sim \mathrm{Re}^{-0.8}$, which is smaller than the Kolmogorov microscale scaling $\mathrm{Re}^{-0.75}$ by the factor $\mathrm{Re}^{-0.05}$, or $R_\lambda^{-0.1}$. For experimentally accessible R_λ values (order 10^3), this factor is about one half. The implication is that the experimental conditions permit a detectable degree of small scale anisotropy with only a slight deviation from Kolmogorov microscale scaling because q is close to $\frac{2}{3}$. The possibility that this deviation may be observable (subject to measurement imprecision and other intermittency mechanisms that might cause such a deviation) is an interesting issue for future investigation.

Equation (3.2) implies multiplicative increase of the amplitudes of successive odd-order structure functions. Logarithmic plots of measured structure functions, compensated by dividing by the power of R/L corresponding to the observed scaling, indicate that the ratios of successive amplitudes are in fact constant [1]. In contrast, the data in Figs. 10 and 11 of [1] indicate that the normalized derivative-moment sequence n_j, compensated by the power of Re appearing on the right hand side of Eq. (4.3), does not increase with j by constant geometric increments. In the present context, this is interpreted as an indication of the influence of the cutoff function of Eq. (2.1) on the integration leading to Eq. (4.2).

Comparison of the model to analogous passive-scalar properties would be a useful additional test. The model should be equally applicable to passive-scalar anisotropy, though the numerical values of the scaling exponents and multiplicative prefactors are not necessarily the same as for the velocity statistics. Scalar derivative-moment measurements, which would provide the clearest test of the model, do not yet provide the needed resolution of high-order Re dependencies [6].

Apart from the quantitative results and their evident limitations, the main conclusion of this study is that simple scaling hypotheses concerning the frequency, intensity, and rapidity of processes (of unknown dynamical origin) that create local high-gradient zones can account for the measured Re scalings of odd-order derivative moments, and provide a framework for further investigation of small scale anisotropy and related intermittency properties of high-Re turbulence. There is an evident conceptual linkage between the anisotropic character of derivative moments and structure functions, but the complex processes controlling structure-function scalings hinder accurate quantification of this linkage.

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